*International Journal of Theoretical Physics, Vol. 46, No. 6, June 2007 (© 2007)*
DOI: 10.1007/s10773-006-9289-7

# Dynamics of Minimal Surfaces

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*Received August 2, 2006; accepted September 27, 2006*
*Published Online: December 19, 2006*

The movement of topological or deformable surfaces was shown to create a helicoid surface. The description of dynamics in the parametric space of helicoid introduces a new understanding to quantum mechanics and field theory. The different equations of quantum mechanics can be obtained from a general equation of motion derived from helicoidal dynamics. It was shown that space and time can be transformed into each other in parametric space.

**KEY WORDS:** topological surfaces; minimal surfaces; helicoid; catenoid; quantum.

## 1. INTRODUCTION

The standard model can describe the world of particles but has about twenty free parameters whose values are not understood theoretically. The arbitrary factors are due to arbitrary branching of particles. The standard model cannot be used to make any predictions without the input data of the fundamental particle properties.

The use of matrix integral methods in evaluating string partition function leads to the hierarchy of nonlinear evolutionary equations of KdV type. Since KdV equation has soliton solution, it turns out that solitons in general take the form of higher-dimensional extended objects, i.e. the D-branes. It is believed that different physical systems such as standard model, supersymmetric gauge theories, black holes, etc. can be modeled using D-branes. A fundamental problem of string or D-brane theories is the reduction of high number of dimensions; but the methods developed for the compactification of dimensions is too arbitrary to construct a fundamental theory. For a given space-time dimension D there can occur different string theories depending on the way the perturbations are done. The perturbative expansion around a particular ground state leads to the development of conformal field theory on Riemannian surfaces, which is governed by the Virasoro algebra and its extensions. However, these theories are difficult to construct because of the very large number of fields involved. In recent years it was shown that a lower

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dimensional brane could blow up into a higher dimensional brane in converse to compactification. A IIA superstring can be blown into a tubular D2-brane by placing it in an appropriate (nonsupersymmetric) background (Park *et al.*, 2003). The tubular topology was first considered in the Kaluza-Klein model where the fifth dimension is supposed to have a circular topology as in a garden hose, and the coordinate is periodic, $0 \leq \text{my} \leq 2\pi$, where m has dimensions of mass, and, $x_5 = x_5 + 2\pi$. The five-dimensional Riemannian manifold is assumed to have split up according to $R_5 \rightarrow R_4 \times S_1$. Since the dimension of the fifth dimension is so small, it was also assumed that $\partial_5 \rightarrow 0$. But the dimensional breaking is a problem that needs to be solved and the same problem is still faced in superstring theory. The Kaluza-Klein theory describes an infinite number of four-dimensional fields and also an infinite number of four-dimensional symmetries. However, it was suggested that the universe is a 3-brane in a higher dimensional space-time (Gibbons and Wiltshire, 1987). It is possible to think that the strong, weak, and electromagnetic forces might be confined to the worldvolume of the brane but the gravity propagates all over (Antoniadis, 1990; Witten, 1996; Arkani-Hamid *et al.*, 1998; Dienes *et al.*, 1998).

As the particles are spread out in more dimensions the space-time divergences are reduced but new divergences coming from the increased number of internal degrees of freedom are encountered.

A remarkable property of D-branes in string theory is that a single Dp-brane behaves like a classical p-dimensional surface (Van Raamsdonk, 2002). Surfaces are treated as deformable objects in differential geometry. Strings, though they can stretch, are not assumed to be strictly deformable; they sweep a two-dimensional world sheet as they move in space-time. The world sheet is embedded in higher dimensional space, and the dimensions higher than four-dimensional space-time are believed to be compacted after big-bang.

The degenerate vacua that arise from spontaneous symmetry breaking theories lead to conditions for identifying topological solitons. They are translated into the quantized flux tubes in Type II superconductors. The flux lines are assumed to be confined in a cylindrical tube separated by sharp boundaries from its surroundings, that is, the tube is embedded in surrounding field lines. In the literature, the flux tubes have always been assumed to conserve their shapes; however there is no reason why they should stay rigid, because they have self-interaction terms, which change the surface and thus the dynamics. The deformability was somehow taken into consideration in recent theories which are built up on the anti-de Sitter (AdS) space by introducing a warp factor (Gibbons and Wiltshire, 1987; Antoniadis, 1990; Witten, 1996; Arkani-Hamid *et al.*, 1998; Dienes *et al.*, 1998; Van Raamsdonk, 2002; Randall and Sundrum, 1999a,b), which is a rapidly changing function of an additional dimension. Randall and Sundrum (Randall and Sundrum, 1999a,b) considered a 3-brane, that is, a Minkowski space-time embedded in 5-dimensional anti-de Sitter space-time ($AdS_5$). They found that there exists a massless graviton associated with Newtonian gravity, and massive gravitons associated with Kaluza-Klein modes. The former can be reproduced in a sufficiently low-energy limit. The metric of Randall and Sundrum is not factorizable, but the four-dimensional metric is multiplied by a "warp" factor which is a rapidly changing function of an additional dimension, and it is given by Randall and Sundrum (1999a),

$$ds^2 = e^{-2kr_c|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2 \tag{1}$$

where $k$ is a scale of the order of Planck scale, $x^\mu$ are the coordinates for the four dimensions, while $0 \leq \phi \geq \pi$ is the coordinate of the extra dimension. Its size is determined by the compactification radius $r_c$. The geometry is warped exponentially, and this warp factor ensures that gravity is localized on the brane (Padilla, 2002; Mukhopadhyaya *et al.*, 2002). The gravitational field tensor propagates in the higher dimensional manifold so-called the 'bulk' while the standard model fields are assumed to be confined to the 3-dimensional brane. However, the compactification of the higher spacelike dimensions creates many massive fields on the 3-brane. The creation of such fields follows from the dynamics of branes, and it is in fact necessary to have such fields to achieve causality (Kostelecky and Lehnert, 2001). A theory of gravity in a bulk AdS is dual to a conformal field theory (CFT) on its boundary.

The warp factor can be a function of the number of dimensions of the brane (Ito, 2002), while it may also be expressed in terms of internal coordinates (Cardoso *et al.*, 2003). Kobayashi *et al.* (2002) considered the metric,

$$ds^2 = e^{2\alpha(y)}\gamma_{\mu\nu}dx^\mu dx^\nu + dy^2 \tag{2}$$

and have shown that by solving Einstein's equations $\alpha(y)$ can have the values,

$$\alpha(y) = \begin{cases} \mathrm{y}_0 - |\mathrm{y}| & \text{Poincarebrane} \\ \log\left[\sinh(y_0 - |y|)\right] & \text{deSitterbrane} \\ \log\left[\cosh(y_0 - |y|)\right] & \text{anti-deSitterbrane} \end{cases} \tag{3}$$

The warp factor is thus simplified, and the warp factors for the de Sitter (dS) and AdS branes are simply given by hyperbolic functions. For a metric of the type,

$$ds^2 = a^2(z)(dz^2 + \gamma_{\mu\nu}dx^\mu dx^\nu) \tag{4}$$

a conformal-like coordinate $z$ is defined through the equation,

$$z = \int \frac{dy}{a}, \quad a(\mathrm{z}) = \mathrm{e}^{\alpha(\mathrm{y(z)})} \tag{5}$$

Hyperbolic functions show up in the dimension compactification and mapping operations (Nojiri and Odintsov, 2002), and Kluson *et al.* (2005) showed that in global coordinates AdS background metric takes the form,

$$ds^2 = L^2[-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\phi^2] \tag{6}$$

Poincare and global coordinates are related through the relation,

$$x^1 \pm x^0 = \frac{1}{r}(\sinh \rho \sin \phi \pm \cosh \rho \sin \tau) \tag{7}$$

and

$$r = (\sinh \rho \cos \phi \pm \cosh \rho \cos \tau) \tag{8}$$

Campos studied phase transition on thick Minkowski branes, and found that that a Schrödinger-like equation can be obtained by changing to the coordinate $dz = e^{-\alpha} dr$ (Campos and Campos, 2002).

The space-time between the two 3-branes is simply a slice of an $AdS_5$ geometry. The warp factor introduces a kind of topological deformation to the metric. Although topological considerations have been heavily used in gravitation theories in the past, the extensive use in particle physics is seen in the last decades in relation to string and M-brane theories. A deformation that can be attributed to a string, namely to a superconducting cosmic string is a kind of formation of wiggles or screw like structures. A wiggly string behaves as a superposition of a massive rod and a string with a conical deficit (Pogosian and Vachaspati, 2004). The wiggly string picture was changed to a screwed string with torsion by Ferreira (Ferreira, 2002; Bezerra and Ferreira, 2002). The warp compactification of M-theory on 7-manifold yields a flat 4-dimensional Minkowski space (Klaus and Claus, 2004), but when torsion is involved discrete torsion singularities cannot be geometrically resolved, and a totally distinct compactification is needed (Berenstein and Leigh, 2000). Torsion is some independent characteristics of space-time, and in the Einstein-Cartan theory, with or without matter, torsion does not have dynamics, and therefore can only lead to the contact interaction between spins (Shapiro, 2002).

## 2. DYNAMICS OF CURVES AND SURFACES

As explained above, the warp factor compactification methods developed so far showed that, (i) the metric can be expressed in terms of hyperbolic functions, and it indicates that the resulting surfaces are hyperbolic in nature; (ii) in addition, torsion must play an important role in the dynamics. As the string action is evaluated in terms of the area swept by the string since first introduced by Nambu and Goto, it is essential that the dynamics has to be based on the properties of surfaces (Goddard *et al.*, 1973), and these surfaces must be topologically deformable in order to account for many features of branes. One of the important properties of D-branes is that a single brane behaves geometrically like a classical p-dimensional surface, though collections of many different Dp-branes can exist in completely different configurations (Van Raamsdonk, 2002). However a classical surface can form a good basis to understand the dynamics of branes (Dean *et al.*, 1992; Johnson, 2003). Let us consider a metric for an n-dimensional surface,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + \cdots - x_0^2 = 1 \tag{9}$$

We can make the substitutions,

$$z_i^2 = \sum_{i=1}^{n} x_i^2 - x_0^2, \quad z_3^2 = x_3^2 + x_4^2 + \cdots - x_0^2, \quad \text{etc.} \tag{10}$$

so that Eq. (9) becomes,

$$x_1^2 + x_2^2 + z_3^2 = x^2 + y^2 + z^2 = 1 \tag{11}$$

which represents a hyper sphere.

In order to define a surface we need to use two parameters u and v as seen in Fig. 1. '$u$' is the angle that the radius to the point $P$ on the surface makes with the positive z-axis, and '$v$' denotes the angle, which the plane through the $z$-axis and the point $P$ makes with the xy-plane.

The coordinates of $P$ can be written as,

$$x = f_1(u, v) = \alpha \sin u \cos v, \quad y = f_2(u, v) = \alpha \sin u \sin v,$$
$$z = f_3(u, v) = \alpha \cos u \tag{12}$$

The element of arc length $ds$ is,

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{13}$$

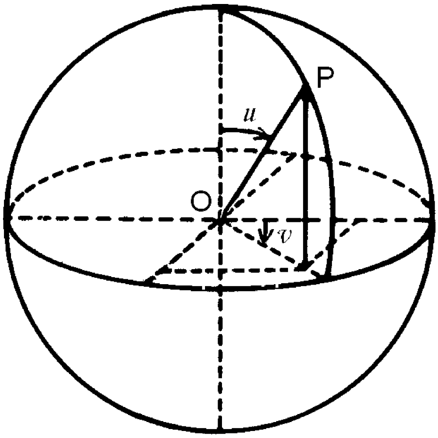


**Fig. 1.** Parametrization of a surface.

where,

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, \quad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv, \quad dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \quad (14)$$

We select any curve with its equation $\phi(u, v) = 0$, so du and dv satisfy,

$$\frac{\partial \phi}{\partial u}du + \frac{\partial \phi}{\partial v}dv = 0 \quad (15)$$

Introducing,

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ F &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\ G &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \end{aligned} \quad (16)$$

Equation (13) then becomes,

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 \quad (17)$$

Changing the notations,

$$E = g_{11}, \quad F = g_{12} = g_{21}, \quad G = g_{22} \quad (18)$$

yields,

$$ds^2 = g_{11}\,du^2 + 2\,g_{12}\,du\,dv + g_{22}\,dv^2 \quad (19)$$

This is a Riemannian metric; so the parameterization of a surface takes us from Euclidean to Riemannian space.

Classical membrane and $p$-brane both are supposed to have a minimum surface area due to minimum action principle. The parameterization of surface needs to obey the minimality principle. The minimal curves on a surface can be found from Eq. (17) by letting $ds^2 = 0$, which gives,

$$Edu^2 + 2F\,du\,dv + G\,dv^2 = 0 \quad (20)$$

This equation defines imaginary curves of double family which lie on a surface. The condition $ds^2 = 0$ implies $EG - F^2 = 0$. Therefore, the minimal lines form an orthogonal system.

The equations of a curve can be given by,

$$x = f_1(u), \quad y = f_2(u), \quad z = f_3(u) \quad (21)$$

The minimality condition requires,

$$f'^2_1 + f'^2_2 + f'^2_3 = 0 \quad (22)$$

This condition can be written in the form,

$$\frac{f_1' + i f_2'}{-f_3'} = \frac{f_3'}{f_1' - i f_2'} = v \tag{23}$$

where $v$ is a constant or function of $u$. Hence, $x$, $y$, and $z$ can be written as,

$$x = \int \frac{1 - v^2}{2} f(u) du, \quad y = i \int \frac{1 + v^2}{2} f(u) du, \quad z = \int v f(u)\, du \tag{24}$$

Consider the case where $v$ is a function of $u$, and take this function of $u$ for a new parameter, and call it $u$ for convenience. Then, Eq. (24) can be written in the form,

$$x = \int \frac{1 - u^2}{2} \mathsf{F}(u) du, \quad y = i \int \frac{1 + u^2}{2} \mathsf{F}(u) du, \quad z = \int u \mathsf{F}(u) du \tag{25}$$

A minimum surface referred to its minimal lines can now be defined in terms of $x$, $y$, and $z$ such that,

$$\begin{aligned} x &= \frac{1}{2} \int (1 - u^2) \mathsf{F}(u) du + \frac{1}{2} \int (1 - v^2) f(v) dv \\ y &= \frac{i}{2} \int (1 + u^2) \mathsf{F}(u) du - \frac{i}{2} \int (1 + v^2) f(v) dv \\ z &= \int u \mathsf{F}(u) du + \int v f(v) dv \end{aligned} \tag{26}$$

where $\mathsf{F}$ and $f$ are any analytic function. This definition is due to Enneper and the details can be found in standard textbooks on differential geometry (Eisenhart, 1960; Do Carmo, 1976; Kreysig, 1991). Equation (26) gives,

$$E = 0, \quad F = \frac{1}{2}(1 + uv)^2\, \mathsf{F}(u) f(v), \quad G = 0 \tag{27}$$

where the linear element is,

$$ds^2 = (1 + uv)^2\, \mathsf{F}(u) f(v) du\, dv \tag{28}$$

Now we can consider a sphere with a unit radius, such that,

$$x^2 + y^2 + z^2 = 1 \tag{29}$$

This equation is valid for both a classical surface where $(x, y, z)$ are Euclidean coordinates or a hyper sphere with n-dimension where '$z$' is given by Eq. (10). Therefore we can study the hyper sphere in a parametric space by simply making the change $(u, v)_{euc.} \rightarrow (u, v)_{hyper}$. Equation (29) can be written as,

$$\frac{x + iy}{1 - z} = \frac{1 + z}{x - iy} = u$$

$$\frac{x - iy}{1 - z} = \frac{1 + z}{x + iy} = v \tag{30}$$

where $u$ and $v$ are respective ratios. It is clear that $u$ and $v$ are conjugate imaginaries. From Eq. (30) one can obtain,

$$x = \frac{u + v}{uv + 1}, \quad y = \frac{i(v - u)}{uv + 1}, \quad z = \frac{uv - 1}{uv + 1} \tag{31}$$

From these relations one gets,

$$ds^2 = \frac{4\, du\, dv}{(1 + uv)^2} \tag{32}$$

If the minimal surface deforms into a new surface the parameters change as $(u, v) \to (u_1, v_1)$, and $\mathsf{F}(u) f(v) \to \mathsf{F}_1(u_1) f_1(v_1)$ respectively; and they satisfy,

$$(1 + uv)^2\, \mathsf{F}(u) f(v)\, du\, dv = (1 + u_1 v_1)^2\, \mathsf{F}_1(u_1) f_1(v_1)\, du_1\, dv_1 \tag{33}$$

The equations which provide a correspondence between the two surfaces can be of the form,

$$u_1 = f(u), \quad v_1 = \Phi(v) \quad \text{or} \quad u_1 = f(v), \quad v_1 = \Phi(u) \tag{34}$$

If either set is substituted in Eq. (33) and also the logarithmic derivative is taken with respect to $u$ and $v$, one obtains,

$$\frac{4\, du_1\, dv_1}{(1 + u_1 v_1)^2} = \frac{4\, dudv}{(1 + uv)^2} \tag{35}$$

So the spherical images of the corresponding parts on two surfaces are equal, and can be made to coincide by a rotation of the unit sphere about a diameter. In other words one surface can be displaced in space such that the normals of the two spheres become parallel, and they have the same representation. Thus one gets $u_1 = u$, $v_1 = v$. Equation (33) now becomes,

$$\mathsf{F}(u) f(v) = \mathsf{F}(u_1) f(v_1) \tag{36}$$

It follows that,

$$\mathsf{F}(u_1) = c\, \mathsf{F}(u), \quad f(v_1) = \frac{1}{c} f(v) \tag{37}$$

where '$c$' denotes a constant. If the surfaces are real, $c$ must be of the form $e^{i\alpha}$.

Equation (36) implies that $\mathsf{F}(u) f(v)$ must be a function of $uv$. So we can let,

$$\mathsf{F}(u) = cu^k, \quad f(v) = c_1 v^k \tag{38}$$

where $c$ and $c_1$ are constants. Hence Eq. (26) can be written as,

$$x = \frac{1}{2} c \int (1 - u^2) u^k\, du + \frac{1}{2} c_1 \int (1 - v^2) v^k\, dv$$

$$y = \frac{i}{2}c\int (1+u^2)u^k\,du - \frac{i}{2}c_1\int (1+v^2)v^k\,dv \tag{39}$$
$$z = c\int u^{k+1}\,du + c_1\int v^{k+1}dv$$

When the surface described by Eq. (39) is rotated by an angle of $\alpha$ about the $z$-axis, the coordinates, say $x$-axis, takes the form,

$$\bar{x} = \frac{1}{2}c\int (1-u^2)\bar{u}^k\,du + \frac{1}{2}c_1\int (1-v^2)\bar{v}^k\,dv \tag{40}$$

A rotation is equivalent to replacing $u$ and $v$ by $ue^{i\alpha}$ and $ve^{-i\alpha}$ respectively. So the following relations also exist,

$$\bar{u} = ue^{i\alpha}, \quad \bar{v} = ve^{-i\alpha} \tag{41}$$

The new coordinates are now,

$$\bar{x} = x\,\cos\alpha - y\,\sin\alpha, \quad \bar{y} = x\,\sin\alpha + y\,\cos\alpha, \quad \bar{z} = z \tag{42}$$

The use of these equations in Eq. (39) gives,

$$\bar{\mathsf{F}} = cu^k e^{-i\alpha(k+2)}, \quad \bar{f}(v) = c_1 v^k e^{i\alpha(k+2)} \tag{43}$$

For the substitution of $\bar{u} = u$ and $\bar{v} = v$, the surface $\bar{S}$ is an associate of $S$, unless $k+2=0$; that is, their tangent planes at corresponding points are parallel. For $k+2=0$ case, they are the same surface, and one has,

$$\mathsf{F}(u) = \frac{c}{u^2} \tag{44}$$

In this case, the resulting expressions for the new positions $x_1$, $y_1$, and $z_1$ are,

$$x_1 = x\,\cos\alpha - y\,\sin\alpha, \quad y_1 = x\,\sin\alpha + y\,\cos\alpha, \quad z_1 = z + 2R(i\alpha c) \tag{45}$$

This is an extremely important conclusion. In a continuous motion with deformation, the surface slides over itself with a helicoidal motion. Therefore it is a helicoid. Since the surface is minimal, it is a minimal helicoid.

In some recent papers the importance of helicoid in AdS/CFT correspondence has been emphasized (Janik and Peschanski, 2000, 2002). In fact the motion of strings and branes has to obey the minimality principle. No matter how many dimensions a brane has they can be compiled as given simply by Eq. (10). Thus any area formed between any other coordinates obeys minimality principle and forms a helicoid. In other words helicoidal pattern exists in both the Euclidean/Minkowski and in the higher dimensional surfaces.

Helicoid is a deformable surface in differential geometry and its shape is seen in Fig. 2. Its minimal surface is twisted, and thus it has a helix angle between its windings. In differential geometry the helicoid or other objects can be described

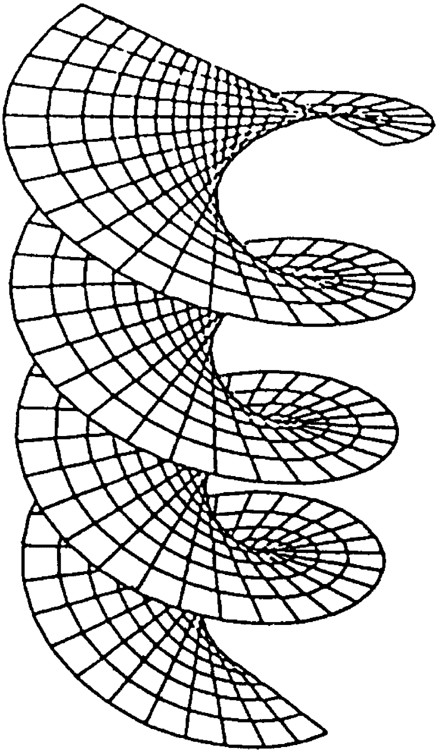

**Fig. 2.** A helicoid (from Dierkes *et al.*, 1992, reproduced with the kind permission of Springer Science and Business Media).

in parametric space of which coordinates are obtained from the parameterization of the ordinary coordinate system.

Each helical curve on the surface of helicoid is like a helical string or rotating string. In string theories the curvature of a closed or rotating string is introduced into the string action as an additional term by utilizing Lagrange multiplier (Johnson, 2003). However the curvature comes out directly from the structure of the helicoid without needing to introduce it as an additional term.

A helicoid is created by drawing a line parallel to the xy-plane through each point of the helix. If the equation of the generating curve at any position is $z = \phi(u)$, the surface generated by the curves is a helicoid, and it admits the following parameterization,

$$\begin{aligned} x &= u \cos v, \quad y = u \sin v, \quad z = \phi(u) + \alpha v, \quad \text{with} \\ 0 &< u < 2\pi \quad \text{and} \quad -\infty < v < \infty \end{aligned} \tag{46}$$

It is rotated about a fixed axis, and at the same time it is translated along this axis. Its translational velocity is proportional to the rotational velocity; the '$\alpha$' denotes the constant ratio of velocities. The translational velocity increases if the rotational velocity increases. The '$\alpha$' is also called the parameter of the helicoidal motion. When $\phi\ (u)$ is a constant, the $u =$ const. curves are straight lines perpendicular to the axis, and the surface is called right helicoid.

A plane and a helicoid are the only two surfaces, which can be generated by the motion of a straight line, i.e. they are ruled surfaces. The helicoid equation given by Eq. (46) is implicitly written as,

$$\bar{x}(\bar{u}, \bar{v}) = (\bar{u} \cos \bar{v}, \bar{u} \sin \bar{v}, \alpha \bar{v}) \tag{47}$$

The parameters can be changed as,

$$\bar{u} = u, \quad \bar{v} = \alpha \ \sinh v, \quad 0 < u < 2\pi \quad \text{and} \quad -\infty < v < \infty \tag{48}$$

This is possible, because, the map is one-to-one, and the Jacobian,

$$\frac{\partial\ (\bar{u}, \bar{v})}{\partial\ (u, v)} = \alpha \cosh u \tag{49}$$

is nonzero everywhere. So the helicoid is expressed by,

$$x(u, v) = (\alpha\ \sinh u\ \sin v,\ \alpha\ \sinh u\ \cos v,\ \alpha\ v) \tag{50}$$

It is easily checked that,

$$E = \alpha^2 \cosh^2 u, \quad F = 0, \quad G = \alpha^2 \cosh^2\ u \tag{51}$$

We also obtain,

$$\mathrm{x}_{uu} + \mathrm{x}_{vv} = 0 \tag{52}$$

This result can be obtained also from the Dirichlet integral (Postnikov, 2001). So the helicoid is a minimal surface. It means that all components of the parameterization are harmonic functions.

Helicoid being a minimal surface in 3-dimensional hyperbolic space can be expressed in four-space as (Fomenko, 1993),

$$\begin{aligned} t_h &= \cosh u\ \cosh v, \quad x_h^1 = \sinh u\ \cos v, \\ x_h^2 &= \sinh u\ \sin v, \quad x_h^3 = \cosh u\ \sinh v \end{aligned} \tag{53}$$

In fact, this is the mapping of ($u$,$v$) plane to hyperbolic space $\mathbf{H}^3$.

It is convenient to use cylindrical coordinates in the study of helicoids. The pseudosphere defined by,

$$t^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 1 \tag{54}$$

is a two-sheeted hyperboloid. In fact, de Sitter space-time with $R > 0$ can also be visualized as the hyperboloid $-v^2 + w^2 + x^2 + y^2 + z^2 = a^2$. The cylindrical coordinates $(r, \phi, z)$ provide the following parameterization for the hyperbolic space,

$$t = \cosh r \cosh z, \quad x^1 = \sinh z \cos\varphi,$$
$$x^2 = \sinh r \sin\varphi, \quad x^3 = \cosh r \sinh z \tag{55}$$

Equation (53) and (55) are of the same form. Thus the helicoid can be considered in two halves; one half can be defined by $\varphi = \alpha z$, while the other half by $\varphi = \pi + \alpha z$ and the line $r = 0$, which is the $z$-axis. The helicoid thus defined is a ruled surface (i.e. a surface that can be generated by the motion of a straight line) stratified into straight lines. The curve $v =$ const. is a straight line in hyperbolic space, and it is the generatrix of the helicoid. The straight line in hyperbolic space is defined as the intersections of two-dimensional planes in Minkowski space.

Another interesting deformable object with a minimal surface is catenoid. Helicoid is the adjoint of catenoid, which is obtained by rotating a catenary described by the curve,

$$x = \alpha \cosh u, \quad z = \alpha u \quad \text{with} \quad -\infty < u < \infty$$

or,

$$r = \alpha \cosh z/\alpha \tag{56}$$

where the last expression is in terms of cylindrical coordinates. A catenoid obtained from the rotation of a catenary has the following parameterization,

$$\mathbf{x}(u, v) = (\alpha \cosh u \cos v, \alpha \cosh u \sin v, \alpha u) \tag{57}$$

It is found for the catenoid that,

$$E = \alpha^2 \cosh^2 u, \quad F = 0, \quad G = \alpha^2 \cosh^2 u \tag{58}$$

Equation (58) is identical to Eq. (51). Therefore it is said that helicoid and catenoid are isometric. Equation (53) can be written for catenoid as,

$$t_c = \sinh u \sinh v, \quad x_c^1 = \cosh u \sin v,$$
$$x_c^2 = \cosh u \cos v, \quad x_c^3 = \sinh u \cosh v \tag{59}$$

The warp factor in *dS* space is in the form of 'sinh' while it is in the form of 'cosh' in AdS space as seen from Eq. (3). This can be understood from the comparison of Eqs. (48) and (56) and also from Eqs. (50) and (57), where 'sinh & cos' substitute for 'cosh & sin' or vice versa. So $dS \leftrightarrow AdS$ transformation can be well understood in terms of helicoid↔catenoid transformation. In fact, as mentioned earlier, the space-time between the two branes is a slice of $AdS_5$ geometry. The two windings of helicoid can behave as two branes connected to each other with certain curvature and torsion, and the properties of an AdS structure is thus displayed.

The harmonic coordinates and the potential terms in different research work also come out to be in either 'sinh' or 'cosh' terms (Hull, 2001; Nojiri and Odintsov, 2002).

Equations (7) and (8) which relate the Poincare and global coordinates in an $AdS_3$ background system are nothing but the coordinates of a helicoid and catenoid. As seen from Eqs. (53) and (59) the terms of Eq. (7) has the same mathematical form of the added terms '$x_h^2 + x_c^1$,' and the terms of Eq. (8) has the same form of '$x_h^1 + x_c^2$.' This is expected because AdS space obeys minimality principle, and helicoid is the basic pattern with surface minimality of the dynamical deformations of n-dimensional space. The wiggly string and the screw string may somehow display a dynamical behavior close to that of helicoid, but they are not real helicoid (Pogosian and Vachaspati, 2004; Ferreira, 2002; Bezerra and Ferreira, 2002).

Catenoid can be deformed into helicoid as seen in Fig. 3.

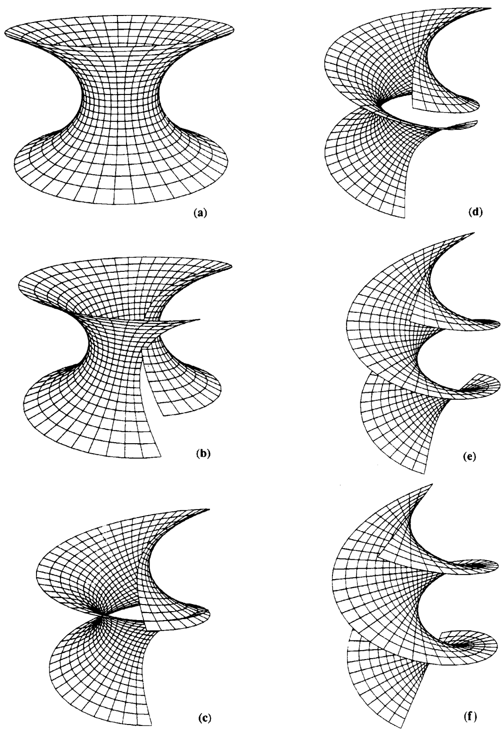


**Fig. 3.** The transformation of catenoid (i.e. Fig. 3a) into helicoid (from Dierkes *et al.,* 1992, reproduced with the kind permission of Springer Science and Business Media).

The spherical or Gauss mapping for a catenoid is a diffeomorphism onto the sphere $\mathbf{S}^2$ minus the north and south poles. It is important to note that catenoid is the only minimal surface of revolution.

It is known that the Gaussian curvature of helicoid ($K_h$) satisfies, $K_h < 1$, therefore it is stable. The Gauss curvature is interpreted as the ratio $dA_N/dA_X$ of the area element of a surface $X$ and of its spherical image $N$ if we take orientation into account. The Gaussian curvature of hyperbolic catenoid ($K_c$) also satisfies this condition therefore it is also stable. However, $K_c$ of spherical catenoid satisfies $K_c > 1$, therefore it is unstable.

Space tearing has been in consideration in string theory in relation to flop-transition in Calabi-Yau shapes. String theorists have strong feeling that space-time can in some way tear. This striking possibility has not been yet established or refuted. It is difficult to visualize tearing of real space-time, but it can easily occur through catenoid-to-helicoid transition in parametric space and also in n-dimensional space beyond Minkowski space, because, parametric space deals with the deformation of the surfaces.

### 2.1. Helicoid and the Sine-Gordon Equation

It is known that the total curvature of a helicoid is constant along a helix. This allows us to define the radius of total curvature "$K$" by,

$$K = -\frac{1}{k^2} \tag{60}$$

where "$k$" is a constant. The negative curvature denotes a pseudospherical surface. Under these conditions $E$, $F$, and $G$ form a set of equations, so-called Codazzi equations which is,

$$R^{\lambda}_{ikl} = \Gamma^{\lambda}_{kl,i} - \Gamma^{\lambda}_{il,k} + \Gamma^{\lambda}_{im}\Gamma^{m}_{kl} - \Gamma^{\lambda}_{km}\Gamma^{m}_{il} \tag{61}$$

or more explicitly,

$$\frac{\partial E}{\partial u} - \frac{\partial G}{\partial u} - 2\frac{\partial F}{\partial v} = 0, \quad \text{and} \quad \frac{\partial E}{\partial v} - \frac{\partial G}{\partial v} - 2\frac{\partial F}{\partial u} = 0 \tag{62}$$

These equations are satisfied by,

$$E = k^2\cos^2\phi, \quad F = 0, \quad G = k^2\sin^2\phi \tag{63}$$

where $\phi$ denotes the angle between the asymptotic lines. The asymptotic lines have directions characterized with zero curvature. When the normal curvature vanishes everywhere, the lines on the surface are called asymptotic curves. When the mean curvature of the surface is zero, the asymptotic lines form an orthogonal set, and they are preserved in a deformation. It must be noted that the asymptotic lines are imaginary on positive curvature surfaces, and real on negative curvature surfaces. They are real for helicoid, which has negative curvature. $\phi$ must satisfy the curvature equation of Gauss given by,

$$K = \frac{1}{2H}\left\{\frac{\partial}{\partial u}\left[\frac{F}{EH}\frac{\partial E}{\partial v} - \frac{1}{H}\frac{\partial G}{\partial u}\right] + \frac{\partial}{\partial v}\left[\frac{2}{H}\frac{\partial F}{\partial u} - \frac{1}{H}\frac{\partial E}{\partial v} - \frac{F}{EH}\frac{\partial E}{\partial u}\right]\right\} \tag{64}$$

where,

$$H = \sqrt{EG - F^2} \tag{65}$$

Since $F = 0$, Eq. (64) reduces to,

$$K = \frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(-\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u}\right) + \frac{\partial}{\partial v}\left(-\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial v}\right)\right] \tag{66}$$

The substitution of Eqs. (60) and (63) in (66) gives,

$$\frac{\partial^2\phi}{\partial u^2} - \frac{\partial^2\phi}{\partial v^2} = \sin\phi \tag{67}$$

which is the Sine-Gordon (SG) equation. This relation shows that helicoid admits soliton solution. The physical ground for this result is that helicoid is a membrane with screw geometry; in addition, it may easily deform through length contraction. That is, it can easily get the shape of a helical membrane, which is a sinusoidal configuration. It is known that two-dimensional ($D = 2$) scalar field theories are capable of giving rise to topological solitons without gauging and without spontaneous symmetry breaking. According to world-sheet field theory, strings of nonzero winding number are topological solitons, and a consistent string theory must include the winding number states. The helicoid structure inherently owns the windings, and the helicoid windings represent the periodic states.

The potential term used in the Lagrangian to obtain SG equation for $D = 2$, 3, and 4 theories are,

$$V_{D=2} = (\alpha/\beta)(1 - \cos\beta\Phi), \quad V_{D=3} = (\lambda/2)(\Phi'\Phi - a^2)^2,$$
$$V_{D=4} = (\lambda/4)(\Phi\Phi - a^2)^2 \tag{68}$$

respectively. The $V_{D=2}$ is a sinusoidal Higgs potential, while $V_{D=4}$ is a Higgs-type potential. Higgs potential automatically yields an infinite number of quasi-degenerate vacua having discrete symmetry. The finite energy solutions satisfy,

$$\Phi(+\infty) - \Phi(-\infty) = 2\pi n/\beta \tag{69}$$

where "$n$" is called winding number. Helicoid is a deformable surface, and it has windings which provide the sinusoidal potential. It also has soliton solution.

For $D > 2$ space-time dimensions the soliton solutions do not exist without introducing additional fields. There are closely related connections between string theory and soliton theory. The conformal field theories lead to the fact that the integrands of string amplitudes are solutions of soliton equations. Although some beautiful mathematical structures are discovered in all such manipulations, the existence of large number of fields does not yield physical results. It seems possible that all aspects of brane-dynamics might be described essentially by helicoidal dynamics, and the soliton solution based on the angle between the asymptotic lines always exists without introducing additional fields provided the extra dimensions can be simply reduced according to Eq. (10).

## 3. REFORMULATION OF QUANTUM MECHANICS

As seen above helicoid admits SG solution. We may search if the fundamental equations of motion can be derived from helicoidal dynamics. At this point we will consider a simple structure at rest, that is, a catenoid membrane. It is known that two D-branes interact by modifying the vacuum fluctuations of stretched open strings. An analogous situation exists in superconducting plates, which attract each other by modifying the vacuum fluctuations of the photon field. Casimir had proposed such attraction long ago in 1940s. For an ordinary quantized particle in a path sum description the Casimir energies originate from closed paths that wind around the compact direction. The Casimir energies in the path sum description are essentially instanton effects. The Hamiltonian H of the translation yields (Polchinski, 1998),

$$H = L_0 + \tilde{L}_0 - \frac{c + \tilde{c}}{24} \tag{70}$$

In a unitary conformal field theory the Virasoro generators at the ground state is $L_0 + \tilde{L}_0 = 0$. So the Casimir energy becomes dependent entirely on the central charge '$c$' and '$\tilde{c}$.' A conserved charge is realized as a contour around a semi-infinite cylinder (Polchinski, 1998). In fact, if there is a central charge, the internal group is Sp(N) while it is $U(N)$ if there are no central charges. The deformable bodies exhibit Sp(N) or similar kind group properties, so we can assign a central charge to deformable objects. Catenoid like geometry was proposed in the past in relation to wormholes connecting different universes. John A. Wheeler advocated the idea that the mouths of wormholes can act somewhat like charged particles (Baez and Muniain, 1994). The solution of a metric for a two dimensional AdS space-time (i.e. $AdS_2 \times S^2$) yields an infinitely long tube or throat like catenoid of topology R $\times$ $S^2$ with fixed radius set by the charge.[2] The conserved charge in a catenoid can be realized as the contour around it, and it creates a capacitive effect on the object. As the catenoid tears into helicoid, the windings are created, and thus the inductive effects are also generated. Therefore the transformation from catenoid to helicoid involves both capacitive and inductive effects, and it involves a net flow or current. However we need to answer to what happens to the electrical charge after catenoid was torn into helicoid. A minimal surface known as 'Scherk's

[2] Johnson (2003), pp. 224–237.

second surface' provides a surface of helicoidal type generated by a screw motion of some planar curve about the $z$-axis. Its $x$- and $y$-coordinates are expressed as the sum of the $x$- and $y$-coordinates of helicoid and catenoid.[3] However a problem exists both in the catenoid torn into helicoid, and in the Scherk's second surface as long as the stability of the system is concerned. They can easily turn back to catenoid. However if we consider a closed helicoid of which front joins its end, then, we get a perfectly toroidal helicoid. This helicoid can be stable but is neutral, that is, it does not carry any charge since its ends are one-to-one joined. In addition, it is closed with no internal topological distortion, therefore it behaves as a superconducting toroidal helicoid. A charge carrying toroidal helicoid must have also a catenoidal surface. This can be achieved only when the front of the helicoid joins its end in a twisted form like a Möbius object, which has Sp(N) or similar kind group. Thus the closed helicoid with Möbius topology (i) is a folded space, (ii) has charge because it also has a partial catenoid surface assembled on helicoid, and (iii) has windings. The features of this structure will be treated soon, but first the dynamics of this membrane will be examined.

The capacitive effect due to charge and the inductive effect due to windings must be considered in the general sense, not strictly in the electromagnetic sense. We can assign the following general attributes to study the motion of the Möbius helicoid under a potential.

$\varphi$: potential
$C$: general capacitance of the membrane
$L$: general inductance of the membrane
$R$: general resistance of the membrane
$G$: leakage from/to the surroundings in contact

During deformation the disturbance created at the surroundings can be realized as the leakage term, and the effect of surroundings onto the system is also a leakage term as well. As the surface area changes during transformation the stress on the object also changes. Since the minimality principle requires equilibrium with the surroundings some amount of energy must be exchanged with the surroundings, and this will be represented by a '$G$' term.

In a helicoid the change in potential and in current $\Delta j$ on a distance $\Delta x$ can be given by,

$$\Delta\varphi = -R\Delta x j(x_1, t) - L\Delta x \frac{\partial j(x_1, t)}{\partial t} \tag{71}$$

$$\Delta j = G\Delta x \varphi(x_1, t) - C\Delta x \frac{\partial \varphi(x_1, t)}{\partial t} \tag{72}$$

[3] Dierkes *et al.* (1992), pp. 140–144.

where $x < x_1 < x + \Delta x_1$. Dividing by $\Delta x$, and taking the limits as $\Delta x \to 0$ yields,

$$\frac{\partial \varphi}{\partial x} = -Rj(x,t) - L\frac{\partial j(x,t)}{\partial t} \tag{73}$$

$$\frac{\partial j}{\partial x} = G\varphi - C\frac{\partial \varphi}{\partial t} \tag{74}$$

The deformations of the membrane cause distortions along radial and axial directions. It is known from electromagnetic theory that four-vector potential $A_\mu$ transforms as

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \tag{75}$$

where $\alpha(x)$ is a function of the space-time coordinates. This is the simplest gauge form. The deformation of catenoid into helicoid creates a strain and thus creates helicoid windings. Therefore $L$ changes as,

$$L \to L + \ell\frac{d}{dx} \tag{76}$$

and similarly $C$ also changes as,

$$C \to C + c'\frac{d}{dt} \tag{77}$$

The substitution of Eqs. (76) and (77) in Eqs. (73) and (74) gives,

$$\frac{\partial \varphi}{\partial x} = -Rj - L\frac{\partial j}{\partial t} - \ell\frac{\partial^2 j}{\partial x \partial t} \tag{78}$$

$$\frac{\partial j}{\partial x} = G\varphi - C\frac{\partial \varphi}{\partial t} - c'\frac{\partial^2 \varphi}{\partial t^2} \tag{79}$$

These two equations can be combined by eliminating '$j$.' One then gets,

$$\begin{aligned} Rc'\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + RC\frac{\partial \varphi}{\partial t} - LG\frac{\partial \varphi}{\partial t} + LC\frac{\partial^2 \varphi}{\partial x \partial t} - \ell G\frac{\partial^2 \varphi}{\partial x \partial t} \\ + Lc'\frac{\partial^3 \varphi}{\partial t^2 \partial x} + \ell C\frac{\partial^3 \varphi}{\partial x^2 \partial t} + \ell c'\frac{\partial^4 \varphi}{\partial x^2 \partial t^2} - RG\varphi = 0 \end{aligned} \tag{80}$$

The windings of helicoid are constant in number, that is, they are quantized, similar to the flux lines in a flux tube, because, the Möbius helicoid is a closed object. In this respect, $R$, $c'$, and $\ell$ must have such values that the changes taking place according to Eq. (80) must be quantized.

By letting,

$$R = \frac{2m}{\hbar^2}, \quad C = \frac{\hbar}{i}, \quad c' = \frac{1}{Rc^2} = \frac{\hbar^2}{2mc^2} \tag{81}$$

where '$c$' is light velocity; these equations can now be substituted in Eq. (80). We can observe the following special cases.

(i) If there are no leakage (i.e. $G$ terms $\rightarrow 0$, or simply $G = 0$), no inherent inductive and capacitive effects (i.e. $L = 0$, and $C = 0$), and no twist deformation (or induced inductance or strain i.e. $\ell = 0$), but there is only an induced capacitance due to distortion (i.e. $c' \neq 0$), then Eq. (80) becomes,

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} = 0 \tag{82}$$

which is the electromagnetic wave equation without a source term.

(ii) If there are no leakage (i.e. $G = 0$), and no twist deformation ($L = \ell = 0$), but there is an inherent capacitive deformation (i.e. $C \neq 0$) and it is not time dependent, that is, if the time dependent deformation is not nonlinear, (i.e, $c'd/dt \rightarrow 0$, or simply $c' = 0$), then Eq. (80) gives,

$$-i\frac{\partial\varphi}{\partial t} = -\frac{\hbar}{2m}\frac{\partial^2\varphi}{\partial x^2} \tag{83}$$

which is the Schrödinger equation without the potential term. We may introduce a potential term to take into account the effect of environment on the motion of a particle.

(iii) If $L = 0$, $C = 0$, $\ell = 0$, but $c' \neq 0$, then Eq. (80) gives,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = \frac{2m}{\hbar^2}G\varphi \tag{84}$$

or,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = \kappa^2\varphi \tag{85}$$

with $\kappa = mc/\hbar$ for the reciprocal of the Compton wavelength provided,

$$G = \frac{mc^2}{2} \tag{86}$$

Equation (85) is the Klein-Gordon (KG) equation. In relativistic description, a particle absorbs energy from an externally applied force, and uses it to accelerate itself (i.e. changes it into translation energy) and also to increase its mass, hence $G \rightarrow -G$.

The leakage '$G$' is associated directly with mass as seen from Eq. (86). As mentioned above a closed helicoid without a folded space (i.e. without a catenoidal surface) behaves like a superconducting helicoidal toroid, and it does not need to exchange anything with its surroundings. The exchange with the surroundings (i.e. leakage) becomes possible in a Möbius helicoid, because, it has a folded space. In other words, the existence of catenoid provides a permanent folded space on the Möbius helicoid. So 'mass' can be realized as 'folded space.'

(iv) If $L = 0$, $C = 0$, $\ell \neq 0$, $c' \neq 0$, and $G = 0$,

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} + \ell\frac{\hbar^2}{2mc^2}\frac{\partial^4\varphi}{\partial x^2\partial t^2} = 0 \tag{87}$$

(v) If $L = 0$, $C = 0$, $\ell \neq 0$, $c' \neq 0$, and $G \neq 0$,

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} + \ell\frac{\hbar^2}{2mc^2}\frac{\partial^4\varphi}{\partial x^2\partial t^2} + \ell\frac{mc^2}{2}\frac{\partial^2\varphi}{\partial x\partial t} = \frac{m^2c^2}{\hbar^2}\varphi \tag{88}$$

This last equation is different from the KG equation by the third and fourth terms involving the strain of the helicoidal membrane. It is seen that in all cases (i.e. (i)–(v)) we have $L = 0$. The membrane does not own an inherent winding and it may be created only after distortion. In other words, the undisturbed space does not have a twist or helicoid structure. Since $\ell = 0$ in Eqs. (82), (83), and (85), all the known wave equations (i.e. electromagnetic, Schrödinger, and KG) do not have strain terms. Equation (88) is the most general among all these equations. We may think of linearizing it in time by following the Dirac's procedure of linearizing the KG equation. However, this procedure does not work here, because of the mixed derivative (i.e. $\partial^2\varphi/\partial x\partial t$ and $\partial^4\varphi/\partial x^2\partial t^2$) terms. Here, space and time are fused up not as in 4-dimensional Minkowski space-time, but rather as in curved space with torsion, because, the object is deformable and we have mixed derivatives; and both helicoid and catenoid have Riemannian metric as mentioned above. Therefore Dirac's procedure of linearization of the KG equation is a special and simplified case. The true physics must have the mixed derivatives as in Eq. (88).

### 3.1. Space and Time Transformation:

A linear helicoid is not stable as discussed above; it has to move with its energy or decay by loosing energy. As mentioned earlier it can stabilize only when it formed a closed loop or toroidal helicoid. In Möbius helicoid the inner and the outer surfaces are continuously interchanged due to Möbius topology. In other words time and space coordinates are unified and interchangeable.

Now let us consider Eqs. (53) and (59) which give the coordinates of helicoid and catenoid respectively. The derivatives of $t$ and $x^3$ terms of Eqs. (59) and (53) with respect to u yield the following relations.

$$\partial_u\left(x_c^3\right) = t_h, \quad \partial_u\left(x_h^3\right) = t_c \tag{89}$$

$$\partial_u(t_c) = x_h^3, \quad \partial_u(t_h) = x_c^3 \tag{90}$$

The derivative of the axial coordinate of catenoid (i.e. $x_c^3$) gives time coordinate of helicoid (i.e. $t_h = x_h^1$), or vice versa; that is, they can be interchanged. Similarly,

$$\partial_u\left(x_c^1\right) = x_h^2, \quad \partial_u\left(x_c^2\right) = x_h^1 \quad \text{and} \quad \partial_u\left(x_h^1\right) = x_c^2, \quad \partial_u\left(x_h^2\right) = x_c^1 \tag{91}$$

Equation (53) satisfies,

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - t^2 = -1 \tag{92}$$

which is Eq. (54). However Eq. (59) satisfies,

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - t^2 = 1 \tag{93}$$

which is the Lorentz metric for $ds^2 \to 1$. The catenoid $\leftrightarrow$ helicoid transformation is actually a transformation of metrics given by Eqs. (92) and (93) to each other. This is nothing but a transformation of Lorentzian and four dimensional Euclidean metrics to each other.

## 4. CONCLUSIONS

The motion of topological or deformable surfaces generates a helicoid. The fundamental equations of quantum mechanics can be obtained from the dynamics of a helicoidal (Möbius helicoid) surface. The transformation of helicoid and catenoid into each other brings in new understanding of some of the basic concepts such as space-to-time or vice versa transformation.

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